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HYDROELASTIC BEHAVIOR OF A COMPLEX STRUCTURE FLOATING ON WAVES

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A numerical method is developed to solve the plane problem of the hydroelastic behavior of a complex structure floating on the surface of an ideal incompressible fluid of finite depth. The motion of the structure described by a deflection function is considered steady-state under the action of incident waves. The hydrodynamic part of the problem is solved using the proposed approach based on the normal-mode method for homogeneous plates. The problem is reduced to a system of linear algebraic equations by means of a transition matrix between representations of the required deflection in the form of expansion in the vibration eigenfunctions of the structure and the plate. It is shown that the results of the calculation performed are in good agreement with available calculation results for a two-part hinged structure at wavelengths comparable to the length of the structure.

Key words: *hydroelasticity, floating structure, elastic inhomogeneous beam, eigenfunctions, hydrodynamic pressure, normal-mode method.*

The hydroelastic behavior of bodies floating on the fluid surface is of significant practical interest due to the application of such objects as floating platforms, tankers, pontoon structures, etc. A feature of such structures is that their length far exceeds the transverse dimensions; therefore, the design of such bodies is modeled using a thin elastic plate. Because the mechanism of interaction of the fluid and structure is difficult to study, the behavior of floating elastic plates has usually been studied under the assumption that the plate is homogeneous. In practice, however, such structures are inhomogeneous and multilayered. Under these conditions, the mathematical modeling of complex structures and development of methods for studying their hydroelastic behavior on waves assume great importance.

In the present paper, we consider the plane linear problem of the behavior of an arbitrary structure on waves. The fluid layer of finite depth is bounded from below by an impenetrable bottom. The thickness of the structure is much smaller than its length. The draft of the structure is ignored. The motion of the structure due to the interaction with a plane surface wave of small amplitude is considered steady-state. It is assumed that periodic vibrations of the structure occur at a frequency equal to the frequency of the incident wave.

External loads on the structure are exerted only by the fluid. The elastic behavior of the structure under distributed load is described by a system of differential equations dependent on the employed model of an elastic body. At the same time, the hydrodynamic-pressure distribution over the wetted part of the structure depends on its deflection and is determined from the solution of the problem of diffraction of a plane surface wave on a floating elastic body. Thus, the hydrodynamic and elastic parts of the problem are coupled and should be solved simultaneously.

The problem of surface-wave diffraction on an isotropic elastic plate was solved in [1]. The problems for homogeneous thin structures are solved using the normal-mode method. The deflection of a homogeneous plate is represented as a superposition of its free vibration modes in vacuum. For a homogeneous beam with the free ends,

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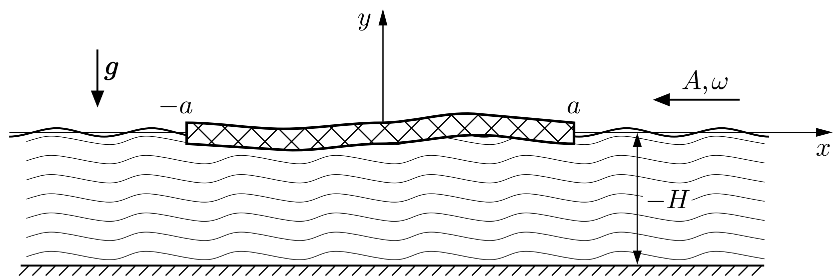

Fig. 1. Diagram of motion of the structure and fluid.

its free vibration modes are known and are represented in analytical form, these eigenmodes being independent on the elastic properties of the plate. The solution of the hydrodynamic part of the problem with specified plate deflection functions yields hydrodynamic coefficients that also do not depend on the plate parameters and are used to determined the hydrodynamic pressure on the plate. Because only the wetted part of the floating complex structure is in contact with the fluid (as in the case with plate), the hydrodynamic part of the general problem remains unchanged, and the elastic part of the problem depends on the employed mathematical model of an elastic body.

In the present paper, an arbitrary floating structure is modeled by an Euler inhomogeneous beam with coefficients dependent on the longitudinal coordinate. The ends of the beam are free of stress. The required function is the vertical deflection of the beam. The purpose of the present work is to extend the normal-mode method to the case of floating inhomogeneous structures.

Formulation of the Problem. The hydroelastic behavior of a floating inhomogeneous structure is studied using linear theory. The draft and thickness of the structure are assumed to be small compared to its length $2a$ and depth H. Time-periodic vibrations of the structure are caused by a surface wave of small amplitude A and frequency ω which is incident from the right (Fig. 1). The origin of the Cartesian rectangular coordinates xy is at the middle of the structure. It is assumed that the fluid is ideal, heavy, and incompressible and its flow is two-dimensional and potential. The fluid layer ($-H \leqslant y \leqslant 0$) is bounded from below by a horizontal impenetrable bottom ($y = -H$). The regions of the upper boundary of the fluid $x > a$, $x < -a$ correspond to the free surface of the fluid layer, and the region $-a < x < a$ to the area of contact of the structure with the fluid. The functions $\eta(x, t)$ and $w(x, t)$ describe the behavior of the free surface and the normal deflection of the lower surface of the structure, respectively:

$$y = \begin{cases} \eta(x,t), & |x| > a, \\ w(x,t), & |x| < a \end{cases}$$

(t is time). In the linear formulation, the fluid flow is described by the velocity potential $\varphi(x, y, t)$. Below, nondimensional variables are used. The plate half-length a is taken as the length scale, $1/\omega$ as the time scale, $\rho g A$ as the hydrodynamic-pressure scale (ρ is the fluid density, and g is the acceleration due to gravity), $A\omega a$ as the velocity-potential scale, and A ($x = ax'$, $y = ay'$, $t = t'/\omega$, $\eta = A\eta'$, $w = Aw'$, $\varphi = A\omega a\varphi'$, and $p = \rho g A p'$) as the scales of the strain of the structure and free-surface elevation. The equations of motion for the fluid and the boundary conditions in the dimensionless variables are written as

$$\varphi_{xx} + \varphi_{yy} = 0 \qquad (-\infty < x < \infty, \quad -H_0 < y < 0),$$

$$\varphi_y = 0 \qquad (-\infty < x < \infty, \quad y = -H_0), \tag{1}$$

$$\varphi_y = \begin{cases} \eta_t, & y = 0,\ |x| > 1, \\ w_t, & y = 0,\ |x| < 1, \end{cases} \qquad \gamma\varphi_t = \begin{cases} -\eta, & y = 0,\ |x| > 1, \\ -p - w, & y = 0,\ |x| < 1, \end{cases}$$

where $H_0 = H/a$ and $\gamma = a\omega^2/g$; primes are omitted everywhere. We note that, in the dimensionless variables, the frequency and amplitude of the wave incident on the structure are equal to unity.

We shall solve the problem imposing the following radiation conditions on the behavior of the fluid free surface for $x \to \pm\infty$:

$$\eta(x,0,t) \sim \cos(kx+t) + A^+ \cos(kx - t + \delta^+) \qquad (x \to +\infty),$$
$$\eta(x,0,t) \sim \cos(kx+t) + A^- \cos(kx + t + \delta^-) \qquad (x \to -\infty). \tag{2}$$

Here A^+ and A^- are the amplitudes of the reflected and transmitted waves, respectively; δ^+ and δ^- are the corresponding phase shifts; k is the dimensionless wavenumber that is a positive solution of the equation $k \tanh kH_0 = \gamma$. The quantities A^+, A^-, δ^+, and δ^- are not known beforehand and must be determine during the solution of the complete problem.

Under the assumption of periodic vibrations of the structure, the velocity potential $\varphi(x,y,t)$ corresponding to steady-state motion, the normal deflection of the lower surface of the structure $w(x,t)$, and the hydrodynamic pressure $p(x,y,t)$ are represented as [1]

$$\varphi(x,y,t) = \varphi_\Pi(x,y,t) + \mathrm{Re}\,(i\,\mathrm{e}^{it}\,\Phi(x,y)), \qquad w(t,x) = \mathrm{Re}\,(\mathrm{e}^{it}\,W(x)),$$
$$\varphi_\Pi(x,y,t) = -\frac{1}{\gamma}\,\frac{\cosh\,(k[y+H_0])}{\cosh\,(kH_0)}\,\sin(kx+t), \qquad p(x,y,t) = \mathrm{Re}\,(\mathrm{e}^{it}\,P(x,y)), \tag{3}$$

where φ_Π is the velocity potential of the incident wave in the absence of the structure. The functions $\Phi(x,y)$, $W(x)$ and $P(x,y)$ are complex. Then, the hydrodynamic part of the basic problem (1) becomes

$$\Phi_{xx} + \Phi_{yy} = 0 \quad (-\infty < x < \infty,\ -H_0 < y < 0), \qquad \Phi_y = 0 \quad (y = -H_0),$$
$$\Phi_y - \gamma\Phi = \begin{cases} 0, & y = 0,\ |x| > 1, \\ -P(x,0), & y = 0,\ |x| < 1. \end{cases} \tag{4}$$

The radiation conditions (2) for functions (3) become

$$\Phi(x,0) \sim B^+\,\mathrm{e}^{-ikx} \quad (x \to +\infty), \qquad \Phi(x,0) \sim B^-\,\mathrm{e}^{ikx} \quad (x \to -\infty),$$

where B^+ and B^- are unknown coefficients. It should be noted that the fluid flow depends only on the deflection $W(x)$ of that part of the structure which is in direct contact with the fluid and does not depend on the strain of its other parts. The behavior of the elastic structure exposed to waves is determined by the hydrodynamic-pressure distribution $P(x,0)$ which depends on the deflection $W(x)$ according to the Cauchy–Lagrange integral:

$$P(x,0) = \gamma\Phi(x,0) - W(x) + \mathrm{e}^{ikx}\,. \tag{5}$$

As a model of the elastic structure we use an Euler inhomogeneous beam whose motion is described by the equation

$$m(x)\,\frac{d^2w}{dt^2} + \frac{d^2}{dx^2}\Big(D(x)\,\frac{d^2w}{dx^2}\Big) = p(x,0,t) \qquad (|x| < a),$$

where $m(x)$ is the mass weight of the beam per unit length, $D(x) = E(x)J(x)$ is the distribution of the bending stiffness along the beam, $E(x)$ is Young's modulus, and $J(x)$ is the axial moment of inertia of the beam cross section with respect to its midline. In the case of a homogeneous plate, these quantities are constant. This approximation is justified provided that the transverse dimensions of the beam are small compared to its length. The action of the distributed load $p(x,0,t)$ on the beam is determined by the hydrodynamic pressure from the fluid. The boundary conditions of the free ends of the beam are given by

$$\frac{d^2w}{dx^2} = \frac{d^3w}{dx^3} = 0 \quad \text{at} \quad x = \pm a.$$

In the new functions (3), the elastic part of the problem becomes

$$-\alpha(x)\omega^2 W(x) + \frac{d^2}{dx^2}\Big(\beta(x)\frac{d^2W(x)}{dx^2}\Big) = P(x,0), \qquad W''(\pm 1) = W'''(\pm 1) = 0, \tag{6}$$

where $\alpha(x) = m(x)/(\rho g)$ and $\beta(x) = D(x)/(\rho g a^4)$; prime denotes differentiation with respect to the coordinate x.

Method of Solution. To find the relation between $P(x,0)$ and $W(x)$, we use the nontrivial solutions $\psi_n(x)$ $(n \geqslant 1)$ of the spectral problem

$$\frac{d^4\psi_n(x)}{dx^4} = \lambda_n^4\psi_n(x), \qquad \psi''(\pm 1) = \psi'''(\pm 1) = 0,$$

where λ_n are eigennumbers. The functions $\psi_n(x)$ are orthonormalized:

$$\int_{-1}^{1} \psi_n(x)\psi_m(x)\,dx = \delta_{nm}.$$

Here δ_{nm} is Kronecker symbol. The eigennumbers λ_n are determined from the dispersion relation $\tan\lambda_n = (-1)^{n+1}\tanh\lambda_n$, where $n \geqslant 3$ and $\lambda_1 = \lambda_2 = 0$. The functions $\psi_n(x)$ are given in [1] and are in fact eigenmodes of the homogeneous beam with free ends.

The functions $W(x)$ and $P(x,0)$ can be written as expansions over the complete system of orthonormalized functions $\{\psi_n\}_{n=1}^{\infty}$:

$$W(x) = \sum_{n=1}^{\infty} A_n\psi_n(x), \qquad P(x) = \sum_{n=1}^{\infty} P_n\psi_n(x) \tag{7}$$

(A_n and P_n are complex coefficients). In [1], the solution of problem (4) (velocity potential $\Phi(x,y)$ along the wetted part of the structure $y = 0$, $|x| < 1$ was represented as

$$\Phi(x,0) = -\sum_{m=1}^{\infty}\Big(\sum_{n=1}^{\infty} P_nC_{nm}\Big)\psi_m(x), \tag{8}$$

where C_{nm} are complex hydrodynamic coefficients in analytical form which do not depend on the elastic characteristics of the structure. Using expansions (7) and (8) and formulas (5), we obtain

$$P_n + \gamma\sum_{m=1}^{\infty} C_{nm}P_m = -A_n + q_n, \tag{9}$$

where

$$q_n = \int_{-1}^{1} \psi_n(x)\,\mathrm{e}^{ikx}\,dx, \qquad n \geqslant 1.$$

The system of algebraic equations (9) allows one to calculate the coefficients P_n in the expansion of the hydrodynamic pressure (7) if the coefficients A_n in the expansion of the deflection of the structure are known. The coefficients C_{nm} and q_n do not depend on the type of structure.

To determine the unknown coefficients A_n, we represent the deflection $W(x)$ in the form of

$$W(x) = \sum_{m=1}^{\infty} E_mW_m(x), \tag{10}$$

where $W_m(x)$ are nontrivial real solutions of the spectral problem

$$\frac{d^2}{dx^2}\Big(\beta(x)\,\frac{d^2W_m(x)}{dx^2}\Big) = \alpha(x)\omega_m^2W_m(x), \qquad W_m''(\pm1) = W_m'''(\pm1) = 0,$$

i.e., vibration eigenmodes of the structure with stress-free ends in vacuum; ω_n are their corresponding vibration eigenfrequencies. It is easy to see that the system of functions $\{W_n\}_{n=1}^{\infty}$ is generally orthogonal and is considered normalized:

$$\int_{-1}^{1} \alpha(x)W_n(x)W_m(x)\,dx = 0 \quad (n \neq m), \qquad \int_{-1}^{1} \alpha(x)W_n^2(x)\,dx = 1.$$

If the eigenmodes $W_m(x)$ and the eigenfrequencies ω_n of the structures are known, then, substituting the deflection (10) and hydrodynamic pressure (7) into the equation of the elastic line (6), multiplying this equation by $W_m(x)$, and integrating the result with respect to x in the interval $[-1,1]$, we obtain

$$(\omega_m^2 - \omega^2)E_m = \sum_{n=1}^{\infty} P_nS_{mn} \qquad (m \geqslant 1), \tag{11}$$

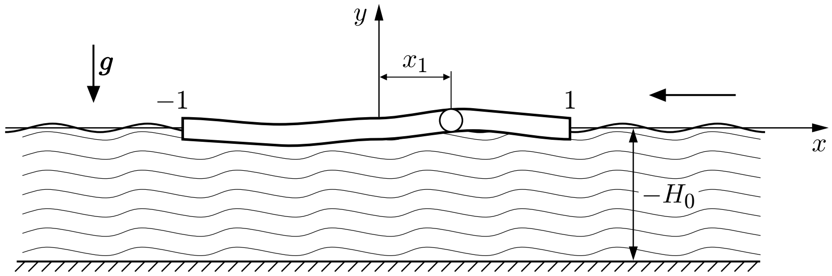

Fig. 2. Two-part plate with a single hinge.

where

$$S_{mn} = \int_{-1}^{1} W_m(x)\psi_n(x)\,dx. \tag{12}$$

Next, multiplying two alternative expressions for the deflection $W(x)$ (7) and (10) by $\psi_n(x)$ and integrating the result with respect to x from -1 to 1 with (11) taken into account, we obtain the following expression for A_n:

$$A_n = \sum_{k=1}^{\infty} U_{nk}P_k, \qquad U_{nk} = \sum_{m=1}^{\infty} \frac{S_{mk}S_{mn}}{\omega_m^2 - \omega^2} \quad (n \geqslant 1). \tag{13}$$

Substitution of representation (13) into (9) yields the following system of linear algebraic equations for the coefficients P_n:

$$P_n + \sum_{m=1}^{\infty} [\gamma C_{nm} + U_{nm}]P_m = q_n \qquad (n \geqslant 1). \tag{14}$$

If any eigenfrequency with number l coincides with the frequency of the incident wave: $\omega_l = \omega$, the coefficients A_n obey the relation

$$A_n = \sum_{\substack{k=1\\k\neq l}}^{\infty} U_{nk}P_k + E_l S_{ln},$$

and instead of (14), one needs to solve the system of equations

$$P_n + \gamma \sum_{m=1}^{\infty} C_{nm}P_m + \sum_{\substack{k=1\\k\neq l}}^{\infty} U_{nk}P_k + E_l S_{ln} = q_n, \qquad \sum_{n=1}^{\infty} P_n S_{mn} = 0.$$

Once the symmetric system of equations (14) is solved, the coefficients E_m can be determined by formula (11) and the beam deflection by formula (10). The hydrodynamic-pressure distribution $P(x, 0)$ on the wetted part of the structure is calculated by formula (7).

Test Calculation. To determine the region of applicability of the proposed method in studies of the hydroelastic behavior of floating structures, we consider a two-part beam with a single hinged connection (Fig. 2). Both parts of the structure are homogeneous plates of different length with identical constant inertial and elastic characteristics: $m(x) = m$ and $D(x) = D$. The hinge connecting the plates is at the point with the dimensionless coordinate x_1, and the total length of the plates is equal to two. The choice of this structure is due to the possibility of analytical representation of its vibration eigenmodes $W_n(x)$ and eigenfrequencies ω_n. In addition, deflections for this type of structure for various incident wavelengths were calculated in [2] using different representations of the required functions $W(x)$ and $P(x, 0)$.

The spectral problem of determining the vibration eigenmodes corresponding to the examined structure is written as

$$\frac{d^4W_n(x)}{dx^4} = \lambda_n^4 W_n(x), \qquad W_n = \begin{cases} W_n^-, & -1 < x < x_1, \\ W_n^+, & x_1 < x < 1, \end{cases}$$

$$(W_n^-)''(-1) = (W_n^-)'''(-1) = 0, \qquad (W_n^+)''(1) = (W_n^+)'''(1) = 0,$$

$$W_n^- = W_n^+, \qquad (W_n^-)'' = (W_n^+)'' = 0, \qquad (W_n^-)''' = (W_n^+)''' \qquad \text{at} \quad x = x_1.$$

Solutions of this problem have the form

$$W_0 = \frac{1}{\sqrt{2}}, \qquad W_1 = \begin{cases} 0, & -1 < x < x_1, \\ \sqrt{3/(1-x_1)^3}\,(x_1 - x), & x_1 < x < 1, \end{cases}$$

$$W_2 = \begin{cases} \sqrt{3/(1+x_1)^3}\,(x_1 - x), & -1 < x < x_1, \\ 0, & x_1 < x < 1, \end{cases}$$

$$W_n^-(x) = B_n\{\sin z_n(x) + \sinh\, z_n(x) - D_n[\cos z_n(x) + \cosh\, z_n(x)]\},$$

$$W_n^+(x) = B_n C_n\{\sin v_n(x) + \sinh\, v_n(x) - F_n[\cos v_n(x) + \cosh\, v_n(x)]\},$$

$$D_n = \frac{\sin z_n^* - \sinh\, z_n^*}{\cos z_n^* - \cosh\, z_n^*}, \qquad F_n = \frac{\sin v_n^* - \sinh\, v_n^*}{\cos v_n^* - \cosh\, v_n^*},$$

$$C_n = \frac{\cosh\, v_n^* - \cos v_n^*}{\cosh\, z_n^* - \cos z_n^*}\,\frac{1 - \cosh\, z_n^* \cos z_n^*}{1 - \cosh\, v_n^* \cos v_n^*}, \qquad B_n = \sqrt{\frac{\lambda_n}{D_n^2 z_n^* - F_n^2 C_N^2 v_n^*}},$$

$$z_n(x) = \lambda_n(x+1), \qquad v_n(x) = \lambda_n(x-1), \qquad z_n^* = z_n(x_1), \qquad v_n^* = v_n(x_1).$$

The eigenvalues λ_n, $n \geqslant 3$ ($\lambda_0 = \lambda_1 = \lambda_2 = 0$) are determined from the equation

$$(\sinh\, z_n^* \cos z_n^* - \cosh\, z_n^* \sin z_n^*)(1 - \cosh\, v_n^* \cos v_n^*)$$
$$= (\sinh\, v_n^* \cos v_n^* - \cosh\, v_n^* \sin v_n^*)(1 - \cosh\, z_n^* \cos z_n^*).$$

It is easy to see that the functions $\{W_n\}_{n=1}^{\infty}$ form a generalized orthogonal system. The eigenfrequencies corresponding to these functions are determined by the formula

$$\omega_n = \frac{\lambda_n^2}{a^2}\sqrt{\frac{D}{m}}.$$

The numerical calculations were performed using the data given in [2] for three incident wavelengths. The half-length of the structure a was set equal to 5 m, and the wavelength was set equal to 0.765 m (short waves), 3.1 m (medium waves), and 8.6 m (long waves). Figure 3 gives calculated deflection of the hinged two-part plate in dimensionless variables versus longitudinal coordinate. The hinge was at a distance $x_1 = 0.5$ from the middle of the beam. The infinite system (14) was solved using the reduction method. It is evident that, in the case of short waves, the calculated curve differs significantly from the test curve over the entire length of the structure (see Fig. 3a and b). As the number of retained modes increases, this difference decreases somewhat (see Fig. 3b). It should be noted that, in the case of short waves, the literature gives different calculation results even for homogeneous plates. In case of medium and long waves, the calculated curve obtained using the proposed method is in good agreement with the test curves for a small number of retained modes (see Fig. 3c and e). The deviation of the calculated curves from the test curves with increasing number of modes (see Fig. 3d and f) is due to inaccurate calculation of the coefficients S_{mn} by formula (12) during integration of rapidly oscillating higher modes. This circumstance is the main limitation of the computation algorithm designed on the basis of the proposed method. However, the examined hinged structure is extreme because the deflection curve has a pronounced break at the hinge point. The indicated limitation is of no great importance for real structures because their eigenfunctions are smoother, which provides the required calculation accuracy in using the minimum number of modes.

The results of the present work show that the proposed generalized normal-mode method can be used to calculate the hydroelastic behavior of complex floating structures at wavelengths comparable to the length of the structure even for a small number of retained eigenmodes of the structure.

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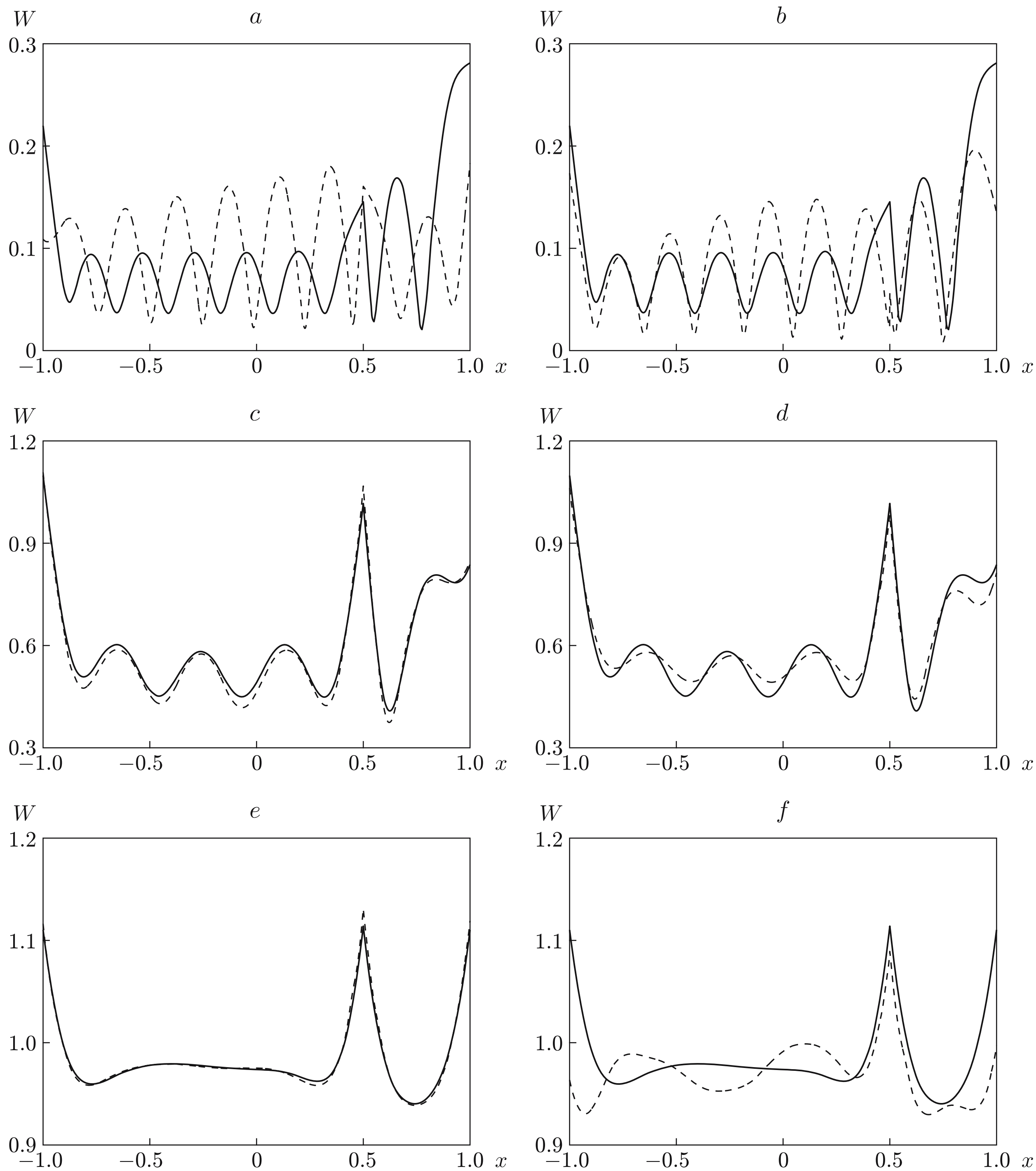

Fig. 3. Deflection of a hinged two-part plate versus longitudinal coordinate: calculation results for 10 retained modes (a, c, and e) and 100 retained modes (c, d, and f); (a, b) short incident waves; (c, d) medium incident waves; (e, f) long incident waves; solid curves are the calculation results of [2], dashed curves are calculation results obtained using the proposed method.

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